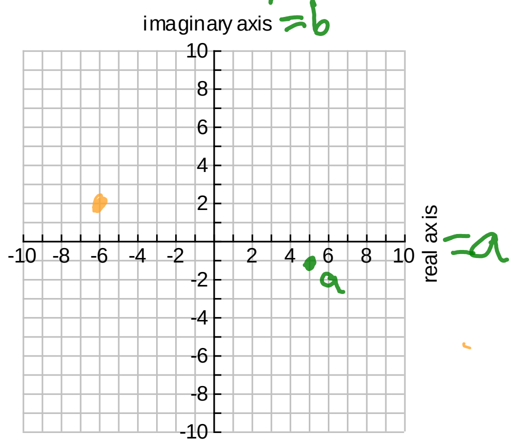
SM 3H

Date: 5/7/24 **Section:** 9.4

Objective: I can graph complex numbers.
I can write trig form of complex numbers.

The complex number $a+bi$ can be thought of as an ordered pair (a,b).

We graph it on the **complex plane** where the horizontal axis is called the real axis and the vertical axis is called the imaginary axis.



Absolute Value or Modulus: $|a+bi| = \sqrt{a^2+b^2}$. (The distance between the number and the origin on the complex plane.)

Examples: Graph each complex number and find its absolute value.

a) $5-i$ real imag

$\sqrt{5^2+1^2} = \boxed{\sqrt{26}}$

b) $-6+2i$ II

$\sqrt{6^2+2^2} = \sqrt{40}$

$\boxed{2\sqrt{10}}$

Trigonometric Form of a Complex Number

If $z=a+bi$ is a complex number, then the trigonometric form of z is

$$z = r(\cos\theta + i\sin\theta), \text{ sometimes abbreviated } z = r\,\text{cis}\,\theta,$$

where r is called the modulus and θ is called the argument, defined as the angle in standard position whose terminal side contains the point (a,b).

$\theta = \tan^{-1}\left(\frac{b}{a}\right)$

$$r = \sqrt{a^2+b^2}$$
$$a = r\cos\theta \text{ and } b = r\sin\theta.$$

We usually use the smallest possible nonnegative angle for θ.

Examples: Write each complex number in trigonometric form. Express θ in degrees.

a) $-2\sqrt{3}+2i$ QII

$r = \sqrt{(2\sqrt{3})^2+2^2} = 4$

$\theta = \tan^{-1}\left(\frac{2}{-2\sqrt{3}}\right) = 150^\circ$

$z = 4(\cos 150^\circ + i\sin 150^\circ)$

b) $5-4i$ QIV

$r = \sqrt{5^2+4^2} = \sqrt{41}$

$\theta = \tan^{-1}\left(\frac{-4}{5}\right) \approx -38.7^\circ$

$z = \sqrt{41}(\cos 321.34^\circ + i\sin 321.34^\circ)$

Example: Write the complex number $12\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$ in the form $a+bi$.

$12\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$

$\boxed{-6\sqrt{2} + 6\sqrt{2}\,i}$

Product and Quotient of Complex Numbers

If $z_1 = r_1\left(\cos\theta_1 + i\sin\theta_1\right)$, and $z_2 = r_2\left(\cos\theta_2 + i\sin\theta_2\right)$, then

$$z_1z_2 = r_1r_2\left(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right)$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right)$$

Examples: Find the product and quotient using trigonometric form.

$$z_1 = 4\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right),\quad z_2 = 8\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$$

a) Find z_1z_2

$32\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$\frac{3\pi}{12} + \frac{\pi}{12} = \frac{\pi}{3}$

b) Find $\frac{z_1}{z_2}$

$\frac{1}{2}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$\frac{3\pi}{12} - \frac{\pi}{12} = \frac{\pi}{6}$

Find the quotient for each pair of complex numbers, using trigonometric form. Write the answer in standard form for complex numbers.

a) $z_1 = 3 + 4i, \quad z_2 = -5 + 2i$

$r = \sqrt{3^2+4^2} = 5 \qquad r = \sqrt{5^2+2^2} = \sqrt{29}$

$\theta = \tan^{-1}\frac{4}{3} = 53.13° \qquad \theta = \tan^{-1}\left(\frac{2}{-5}\right) \approx -21.8$

$338.20°$

$\frac{5}{\sqrt{29}}\left(\cos(-285.07°) + i\sin(-285.07°)\right)$

$\frac{5}{\sqrt{29}}\left(.26 + i(.97)\right) \approx \boxed{.24 + .90i}$

Complex Conjugates

The conjugate of $r\left(\cos(\theta) + i\sin(\theta)\right)$ is $r\left(\cos(-\theta) + \sin(-\theta)\right)$ $\quad r\left(\cos(\theta) - i\sin(\theta)\right)$

A complex number times its conjugate equals r^2.

Proof: $r\left(\cos\theta + i\sin\theta\right)\cdot r\left(\cos(-\theta) + i\sin(-\theta)\right)$

$= r^2\left(\cos(\theta-\theta) + i\sin(\theta-\theta)\right)$

$= r^2\left(\cos 0 + i\sin 0\right)$

$= r^2(1+0i) = \boxed{r^2}$

Example: Find the product of the following and its conjugate: $6\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$. $= 36$